

# Mandatory representation designs $\text{MRD}(4, k; v)$ with $k \equiv 2 \pmod{3}$

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## Abstract

A mandatory representation design  $\text{MRD}(K; v)$  is a pairwise balanced design on  $v$  points with block sizes from the set  $K$  in which for each  $k \in K$  there is at least one block in the design of size  $k$ . In this paper, we consider MRDs with  $K = \{4, k\}$ , where  $k \equiv 2 \pmod{3}$ ,  $k \geq 5$ , and prove that the necessary conditions for existence are sufficient if  $v \equiv 2 \pmod{3}$  and  $v \geq 18k^2$ , or  $v \equiv 1 \pmod{3}$  and  $v \geq 12k^3$ , or  $v \equiv 0 \pmod{3}$  and  $v \geq 8k^4$ .

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## 1. Introduction

A pairwise balanced design PBD is a pair  $(X, \mathcal{B})$ , where  $X$  is a set of *points* and  $\mathcal{B}$  is a collection of subsets of  $X$  called *blocks*, such that each pair of distinct points from  $X$  occurs in a unique block. A  $\text{PBD}(K; v)$  is a pairwise balanced design on  $v$  points in which each block has size an integer in the set  $K$ . A *mandatory representation design*  $\text{MRD}(K; v)$  is a  $\text{PBD}(K; v)$  in which for each  $k \in K$  there is at least one block in the design of size  $k$ . Necessary conditions for the existence of a  $\text{PBD}(K; v)$  are

$$(v-1) \equiv 0 \pmod{\alpha(K)} \quad \text{and} \quad v(v-1) \equiv 0 \pmod{\beta(K)}, \quad (1)$$

where  $\alpha(K) = \gcd\{k-1 | k \in K\}$  and  $\beta(K) = \gcd\{k(k-1) | k \in K\}$ . In a series of three papers, Wilson [28–30] developed an existence theory for PBDs and proved that the necessary conditions are asymptotically sufficient, that is, there exists a constant  $v_0(K)$  such that a  $\text{PBD}(v, K)$  exists for all  $v \geq v_0(K)$  which satisfy the congruences in (1). The problem is that in general there is no guarantee that every block size occurs in such a PBD. But a result of Colbourn and Rödl [11] implies that the necessary conditions (1) for the existence of an MRD are asymptotically sufficient for finite  $K$ .

Colbourn and Rödl's result guarantees the existence of some MRD with the required block sizes provided that the order  $v$  is sufficiently large, but does not control the number of points in any way. Therefore, one attempts to

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determine the spectrum  $B(K) = \{v : \exists \text{MRD}(K; v)\}$  for given  $K$  as accurately as possible. Mandatory representation designs have been extensively studied by Mendelsohn and Rees [21], Rees [23,24], Grützmüller [19], Grützmüller and Rees [16,18,17], and Ge [12]. In particular, in the case  $K = \{4, k\}$  with  $k \equiv 1 \pmod 3$  we have the following result which is the culmination of the contributions of several authors [7,8,12,18,25–27]. Note that the MRDs in part (i) are equivalent to the embedding of a  $(k, 4, 1)$ -BIBD into a  $(v, 4, 1)$ -BIBD.

**Theorem 1.1.** *Let  $k \equiv 1 \pmod 3$ . There exists a mandatory representation design  $\text{MRD}(\{4, k\}, v)$*

- (i) *if  $k \equiv 1, 4 \pmod{12}$ ,  $v \equiv 1, 4 \pmod{12}$  and  $v \geq 3k + 1$ ; or*
- (ii) *if  $k \equiv 7, 10 \pmod{12}$ ,  $v \equiv 7, 10 \pmod{12}$  and  $v \geq 3k + 1$ ; or*
- (iii) *if  $k \equiv 7, 10 \pmod{12}$ ,  $v \equiv 1, 4 \pmod{12}$  and  $v \geq 4k - 3$ , except possibly when  $(k, v) \in \{(10, 52), (22, 121), (22, 124), (22, 133), (22, 136), (22, 145), (22, 148), (22, 244), (34, 229), (34, 232)\}$ .*

In this paper, we continue to investigate the spectrum for MRDs with  $K = \{4, k\}$  now with  $k \equiv 2 \pmod 3$ . The necessary conditions for such MRDs are as follows (we use the notation  $\lceil x \rceil_{a;b}$  to mean the smallest integer not less than  $x$  which is congruent to  $a$  modulo  $b$  and define  $p(t) = \min\{n > 0 : \text{the complete graph } K_n \text{ contains } t \text{ edge-disjoint } K_4\}$ ).

**Theorem 1.2** ([17, Theorem 1.5]). *Let  $k \equiv 2 \pmod 3$ , and suppose that there exists a mandatory representation design  $\text{MRD}(4, k; v)$ . Then the following conditions hold.*

- (i) *If  $k \equiv 2 \pmod 3$  and  $v \equiv 1 \pmod 3$ , then either  $k \equiv 2$  or  $11 \pmod{12}$ , or  $k \equiv 5$  or  $8 \pmod{12}$  and  $v \equiv 1$  or  $4 \pmod{12}$ ; in either case  $v \geq \frac{1}{3}k(2k + 2)$ .*
- (ii) *If  $k \equiv 2 \pmod 3$  and  $v \equiv 2 \pmod 3$ , then either*
  - (a)  *$k \equiv 5$  or  $8 \pmod{12}$ ,  $v \equiv 5$  or  $8 \pmod{12}$  and  $v \geq kp(t) - 3t$ , where  $t = \lfloor \frac{kq-v}{3} \rfloor$  and  $q = \lceil \frac{v}{k} \rceil_{1;3}$ , or*
  - (b)  *$k \equiv 2$  or  $11 \pmod{12}$  and  $v \geq kp(t) - 3t$ , where  $t = \lfloor \frac{kq-v}{3} \rfloor$ , and  $q = \lceil \frac{v}{k} \rceil_{1;6}$  when  $v \equiv 2$  or  $11 \pmod{12}$  while  $q = \lceil \frac{v}{k} \rceil_{4;6}$  when  $v \equiv 5$  or  $8 \pmod{12}$ , with the possible exceptions  $(k, v) = (11, 113)$  and  $(14, 161)$ .*
- (iii) *If  $k \equiv 2 \pmod 3$  and  $v \equiv 0 \pmod 3$ , then either  $k \equiv 2$  or  $11 \pmod{12}$ , or  $k \equiv 5$  or  $8 \pmod{12}$  and  $v \equiv 0$  or  $9 \pmod{12}$ ; furthermore,*

$$v \geq \begin{cases} \frac{1}{2}k(k+1) & \text{if } k \equiv 2, 8, 17, 23 \pmod{24}, \\ \frac{1}{2}k(k+4) - \frac{3}{2} \left\lfloor \frac{k+4}{5} \right\rfloor & \text{if } k \equiv 5, 11, 14, 20 \pmod{24} \text{ and } 5 \mid (k+4), \\ \frac{1}{2}k(k+4) - \frac{3}{2} \left\lfloor \frac{k}{5} \right\rfloor & \text{if } k \equiv 5, 11, 14, 20 \pmod{24} \text{ and } 5 \nmid (k+4). \end{cases}$$

In Section 2 we will show that the necessary conditions for existence are sufficient whenever  $v \equiv 2 \pmod 3$  and  $v \geq 18k^2$ , or  $v \equiv 1 \pmod 3$  and  $v \geq 12k^3$ , or  $v \equiv 0 \pmod 3$  and  $v \geq 8k^4$ .

In the rest of the introduction, we give some definitions and notations as well as some preliminary results which will be used in what follows. We refer the reader to [5,10] for undefined terms as well as a general overview of design theory.

Fundamental to our constructions are a number of designs which we define now. A *group-divisible design* (GDD) is a triple  $(V, \mathcal{G}, \mathcal{B})$ , where  $V$  is a set of *points*,  $\mathcal{G}$  is a partition of  $V$  into *groups* and  $\mathcal{B}$  is a collection of subsets of  $V$  (called *blocks*) such that any pair of distinct points in  $V$  occurs together either in some group or in exactly one block, but not both. A  $K$ -GDD of type  $g_1^{t_1} g_2^{t_2} \dots g_r^{t_r}$  is a GDD in which each block has size from the set  $K$  and in which there are  $t_i$  groups of size  $g_i$ ,  $i = 1, 2, \dots, r$ . We will denote a  $\{k\}$ -GDD as a  $k$ -GDD.

The following families of 4-GDDs will be very useful for our constructions.

**Lemma 1.3** ([9]). *Let  $t$  and  $u$  be positive integers. Then there exists a 4-GDD of type  $t^u$  if and only if the conditions in the following table are satisfied.*

Existence of 4-GDDs of type $t^u$		
$t$	$u$	Condition
1, 5 mod 6	1, 4 mod 12	
2, 4 mod 6	1 mod 3	$(t, u) \neq (2, 4)$
3 mod 6	0, 1 mod 4	
0 mod 6	none	$u = 1$ or $u \geq 4, (t, u) \neq (6, 4)$

**Lemma 1.4** ([6–8,4,26,27,25]). Let  $t$  and  $u$  be positive integers. Then there exists a 4-GDD of type  $t^1 1^u$  if and only if the conditions in the following table are satisfied.

Existence of 4-GDDs of type $t^1 1^u$		
$t$	$u$	Condition
1, 7 mod 12	0, 3 mod 12	$u \geq 2t + 1$
4, 10 mod 12	0, 9 mod 12	$u \geq 2t + 1$

**Lemma 1.5** ([13, Theorem 5.2(ii)-(iv)]). Let  $g, u$  and  $m$  be positive integers. Then there exists a 4-GDD of type  $g^u m^1$  if the conditions in the following table are satisfied.

Existence of 4-GDDs of type $g^u m^1$			
$g$	$u$	$m$	Condition
1, 5 mod 6	0 mod 12	$g \bmod 3$	$g \neq 11, 17, u \neq 12, 24, 72, 120, 168,$ $u \geq (2m + 3)/g + 1$
2, 4 mod 6	0 mod 3	$g \bmod 3$	$g \neq 2, u \geq 192, u \neq 231, 234, 237,$ $u \geq 2m/g + 1$
3 mod 6	0 mod 4	0 mod 3	$u \neq 8, 12, u \geq (2m + 3)/g + 1$

We proceed with the definition of a type of design called modified group divisible design (also known as grid design or as a particular class of double group divisible designs) which serves as an essential tool in our constructions. Let  $k, g, u$  be positive integers. A *modified group divisible design*  $k$ -MGDD of type  $g^u$  is a quadruple  $(V, \mathcal{G}, \mathcal{H}, \mathcal{B})$ , where  $V$  is a finite set of cardinality  $gu$ ,  $\mathcal{G}$  and  $\mathcal{H}$  are two partitions of  $V$  into parts (*groups* and *holes*) and  $\mathcal{B}$  is a family of subsets (*blocks*) of  $V$  which satisfy the properties:

- (i) if  $G \in \mathcal{G}$ , then  $|G| = g$ ;
- (ii) if  $B \in \mathcal{B}$ , then  $|B| = k$ ;
- (iii) if  $G \in \mathcal{G}$  and  $H \in \mathcal{H}$ , then  $|G \cap H| = 1$ ;
- (iv) every pair of distinct elements of  $V$  occurs either in exactly one block, or exactly one group or one hole, but not both.

Assaf and Wei [2], Ling and Colbourn [20], and Ge, Wang and Wei [14] have completely determined the spectrum of 4-MGDDs as recorded in Lemma 1.6.

**Lemma 1.6.** A modified group divisible design 4-MGDD of type  $g^u$  exists if and only if  $(g - 1)(u - 1) \equiv 0 \pmod{3}$  and  $g, u \geq 4$ , except for  $(g, u) = (6, 4)$ .

## 2. Constructions and results

In this section, we develop the constructions for  $\text{MRD}(\{4, k\}, v)$ s required to prove the main result Theorem 2.28. In order to facilitate this, we state an additional necessary condition which does not influence the asymptotic existence question but is important when considering small orders of  $v$  and useful to structure the paper. Let  $x$  be an arbitrary point and let  $\gamma_k$  denote the number of blocks of size  $k$  which contain  $x$ . Then counting pairs containing  $x$  gives  $3\gamma_4 + (k - 1)\gamma_k = v - 1$ , which reduces for  $k \equiv 2 \pmod{3}$  to  $\gamma_k \equiv v - 1 \pmod{3}$ . It will be convenient to consider these cases in separate subsections, where we will first investigate MRDs with  $\gamma_k \equiv 1 \pmod{3}$ , i.e.  $v \equiv 2 \pmod{3}$ . Then, these MRDs will be used to construct MRDs with  $\gamma_k \equiv 2 \pmod{3}$ , i.e.  $v \equiv 0 \pmod{3}$ . And finally both types of MRDs form the basis for the construction of MRDs with  $\gamma_k \equiv 0 \pmod{3}$ , i.e.  $v \equiv 1 \pmod{3}$ .

### 2.1. $v \equiv 2 \pmod 3$ , $\gamma_k \equiv 1 \pmod 3$

We start with constructing some basic MRDs with  $v \equiv 2 \pmod 3$  from 4-GDDs which will serve as ingredient designs in further constructions.

**Lemma 2.1.** *Let  $k \equiv 2 \pmod 3$ ,  $k \geq 5$ . There is a mandatory representation design  $\text{MRD}(\{4, k\}; ku)$*

- (a.i) *if  $k \equiv 2, 8 \pmod{12}$  and  $u \equiv 1 \pmod 3$ ,  $u \geq 4$ ; or*
- (a.ii) *if  $k \equiv 5, 11 \pmod{12}$  and  $u \equiv 1, 4 \pmod{12}$ ,  $u \geq 4$ .*

*Moreover, there is a mandatory representation design  $\text{MRD}(\{4, k\}; (k-1)u+1)$*

- (b.i) *if  $k \equiv 2, 8 \pmod{12}$  and  $u \equiv 1, 4 \pmod{12}$ ,  $u \geq 4$ ; or*
- (b.ii) *if  $k \equiv 5, 11 \pmod{12}$  and  $u \equiv 1 \pmod 3$ ,  $u \geq 4$ .*

**Proof.** Take a 4-GDD of type  $k^u$  from Lemma 1.3 and consider the groups to be blocks of size  $k$  to obtain the desired  $\text{MRD}(\{4, k\}; ku)$ . Furthermore, adjoin a new point at infinity to a 4-GDD of type  $(k-1)^u$  and replace each group and the infinity point by a block of size  $k$  to produce an  $\text{MRD}(\{4, k\}; (k-1)u+1)$ .  $\square$

Note that in the designs constructed  $v \equiv 2 \pmod 3$  and each point lies on either 1 or  $u \equiv 1 \pmod 3$  blocks of size  $k$ , so the condition  $\gamma_k \equiv v-1 \pmod 3$  is satisfied.

The two constructions following next allow us to construct an infinite sequence of mandatory representation designs from just one ingredient design with the property that if for all points in the ingredient design  $\gamma_k \equiv 1 \pmod 3$ , then also in the resulting  $\text{MRD}$   $\gamma_k \equiv 1 \pmod 3$  holds for each point.

**Construction 2.2.** *Let  $k \equiv 2 \pmod 6$ ,  $k \geq 8$  and suppose there is a  $\text{PBD}(\{4, k\}; m)$  with  $m \equiv 2 \pmod 3$ . Then there is a mandatory representation design  $\text{MRD}(\{4, k\}; v)$  for all  $v \geq 3m+k$  with  $v \equiv m \pmod{3k}$ ,  $v \geq 192k+m$ ,  $v \neq 231k+m$ ,  $234k+m$ ,  $237k+m$ .*

**Proof.** Use a 4-GDD of type  $k^u m^1$  which exists by Lemma 1.5 for all  $u \equiv 0 \pmod 3$ ,  $u \geq 192$ ,  $u \neq 231, 234, 237$ ,  $u \geq 2m/k+1$ , consider groups of size  $k$  to be blocks and fill the group of size  $m$  by the  $\text{PBD}(\{4, k\}; m)$  to produce a  $\text{PBD}(\{4, k\}; v = uk+m)$ . Clearly,  $v \equiv m \pmod{3k}$  and we get a PBD for each such  $v$  with  $v \geq \lceil 2m/k+1 \rceil_{0;3} k+m \geq 3m+k$  with the three exceptions listed. Since there is more than one group in the 4-GDD there are blocks of size 4 and  $k$ . So the resulting PBD is indeed an  $\text{MRD}(\{4, k\}; v)$  as desired.  $\square$

**Construction 2.3.** *Let  $k \equiv 5 \pmod 6$  and suppose there is a  $\text{PBD}(\{4, k\}; m)$  with  $m \equiv 2 \pmod 3$ . Then there is a mandatory representation design  $\text{MRD}(\{4, k\}; v)$  for all  $v \geq 3m+k-3$  with  $v \equiv m \pmod{3(k-1)}$ ,  $v \geq 192(k-1)+m$ ,  $v \neq 231(k-1)+m$ ,  $234(k-1)+m$ ,  $237(k-1)+m$ .*

**Proof.** Take a 4-GDD of type  $(k-1)^u(m-1)^1$  which exists by Lemma 1.5 for all  $u \equiv 0 \pmod 3$ ,  $u \geq 192$ ,  $u \neq 231, 234, 237$ ,  $u \geq 2(m-1)/(k-1)+1$ , adjoin one infinite point and fill in the groups together with the infinite point by blocks of size  $k$  or the  $\text{PBD}(\{4, k\}; m)$  to obtain a  $\text{PBD}(\{4, k\}; v = u(k-1)+m)$ . Obviously,  $v \equiv m \pmod{3(k-1)}$  and  $v \geq \lceil 2(m-1)/(k-1)+1 \rceil_{0;3} (k-1)+m \geq 3m+k-3$ . Again, if  $m$  is relatively small we can have three possible exceptions. Note that the construction method ensures that there are blocks of size 4 and  $k$  and, therefore, the PBD constructed is an MRD.  $\square$

In the following we want to apply Constructions 2.2 and 2.3. If we are able to provide a representative PBD in each possible residue class modulo  $3k$  or  $3(k-1)$ , then we have established the existence of an  $\text{MRD}(\{4, k\}; v)$  for all  $v \geq 3m_{\max}+k$  or  $v \geq 3m_{\max}+k-3$ , where  $m_{\max}$  is the number of points in the largest representative PBD. To be more precise, if  $k \equiv 2 \pmod{12}$  we need according to the necessary conditions one representative  $\text{PBD}(\{4, k\}; m_t)$  with  $m_t \equiv 3t+2 \pmod{3k}$  for each  $t = 0, 1, \dots, k-1$ . If  $k \equiv 11 \pmod{12}$ , then a representative  $\text{PBD}(\{4, k\}; m_t)$  with  $m_t \equiv 3t+2 \pmod{3(k-1)}$  is required for each  $t = 0, 1, \dots, k-2$ . Moreover, if  $k \equiv 5 \pmod{12}$  we want representative  $\text{PBD}(\{4, k\}; m_t)$ s with  $m_t \equiv 12t+5, 12t+8 \pmod{3(k-1)}$  for each  $t = 0, 1, \dots, (k-1)/4-1$ , and finally if  $k \equiv 8 \pmod{12}$  we need representative  $\text{PBD}(\{4, k\}; m_t)$ s with  $m_t \equiv 12t+5, 12t+8 \pmod{3k}$  for each  $t = 0, 1, \dots, k/4-1$ . In the next lemmas we will provide these representative PBDs and compute the corresponding bounds for  $v$ .

**Lemma 2.4.** *Let  $k \equiv 2 \pmod{6}$  and  $v \equiv 2 \pmod{3}$ . There exists a mandatory representation design  $\text{MRD}(\{4, k\}; v)$*

- (i) *if  $k \equiv 2 \pmod{12}$ ,  $k \geq 26$ ,  $v \equiv 2 \pmod{3}$  and  $v \geq 18k^2 - 41k + 27$ ; or*
- (ii) *if  $k \equiv 8 \pmod{12}$ ,  $k \geq 44$ ,  $v \equiv 5, 8 \pmod{12}$  and  $v \geq 9k^2 - 32k + 27$ .*

**Proof.** We take as representative designs  $\text{MRD}(\{4, k\}; m_s = (k-1)u_s + 1)$  which exist by Lemma 2.1(b.i) for all  $u_s = 12s + a$ , where  $s \in \mathbb{N}$ ,  $a \in \{1, 4\}$  and  $(s, a) \neq (0, 1)$ . In the latter case we use as representative PBD just a block of size  $k$ . Then

$$m_s = (k-1)(12s + a) + 1 \equiv -12s + k - a + 1 \pmod{3k}.$$

If  $k \equiv 2 \pmod{12}$ , then  $\gcd(12, 3k) = 6$ , and thus with  $a = 1$  and  $s = 0, 1, \dots, k/2 - 1$  we get all residues modulo  $3k$  which are congruent to  $2 \pmod{6}$ . Moreover, with  $a = 4$  and  $s = 0, 1, \dots, k/2 - 1$  we get all residues modulo  $3k$  which are congruent to  $5 \pmod{6}$ . Therefore, we obtained one representative design in which the number of points is congruent to  $3t + 2 \pmod{3k}$  for each  $t = 0, 1, \dots, k - 1$ . The largest representative design has order

$$m_{\max} = (k-1)(12s_{\max} + 4) + 1 = (k-1)(12(k/2 - 1) + 4) + 1 = 6k^2 - 14k + 9.$$

Hence using Construction 2.2 establishes the bound  $v \geq 3m_{\max} + k$  in Case (i). Note that the exceptional cases  $v \neq 231k + m, 234k + m, 237k + m$  do not affect the bound in general as  $3m_{\max} + k > 237k + m_{\max}$  if  $k \geq 26$ .

Similarly, for  $k \equiv 8 \pmod{12}$  with  $a = 1$  or  $4$  and  $s = 0, 1, \dots, k/4 - 1$  we get all residues modulo  $3k$  which are congruent to  $5$  or  $8 \pmod{12}$ . Again using these representative MRDs( $\{4, k\}, m_s$ ) with

$$m_{\max} = (k-1)(12s_{\max} + 4) + a_{\max} = (k-1)(12(k/4 - 1) + 4) + 1 = 3k^2 - 11k + 9$$

in Construction 2.2 yields the bound in Case (ii). It is easily checked that  $3m_{\max} + k > 237k + m_{\max}$  if  $k \geq 44$ , so the exceptional cases listed in Construction 2.2 do not apply.  $\square$

**Lemma 2.5.** *Let  $k \equiv 5 \pmod{6}$  and  $v \equiv 2 \pmod{3}$ . There exists a mandatory representation design  $\text{MRD}(\{4, k\}; v)$*

- (i) *if  $k \equiv 5 \pmod{12}$ ,  $k \geq 53$ ,  $v \equiv 5, 8 \pmod{12}$  and  $v \geq 9k^2 - 32k - 3$ ; or*
- (ii) *if  $k \equiv 11 \pmod{12}$ ,  $k \geq 23$ ,  $v \equiv 2 \pmod{3}$  and  $v \geq 18k^2 - 41k - 3$ .*

**Proof.** Here, we use representative designs  $\text{MRDs}(\{4, k\}; m_s = ku_s)$  which exist by Lemma 2.1(a.ii) for all  $u_s = 12s + a$ , where  $s \in \mathbb{N}$ ,  $a \in \{1, 4\}$  and  $(s, a) \neq (0, 1)$  or a representative PBD( $\{4, k\}; k$ ). Then

$$m_s = k(12s + a) \equiv 12s + ak \pmod{3(k-1)}.$$

If  $k \equiv 11 \pmod{12}$ , then  $\gcd(12, 3(k-1)) = 6$  and thus with  $a = 1$  and  $s = 0, 1, \dots, (k-1)/2 - 1$  we get all residues modulo  $3(k-1)$  which are congruent to  $5 \pmod{6}$ . Moreover, with  $a = 4$  and  $s = 0, 1, \dots, (k-1)/2 - 1$  we get all residues modulo  $3(k-1)$  which are congruent to  $2 \pmod{6}$ . Therefore, we obtained a representative design in which the number of points is congruent to  $3t + 2 \pmod{3(k-1)}$  for each  $t = 0, 1, \dots, k - 2$ . The largest representative design has order

$$m_{\max} = k(12s_{\max} + 4) = k(12((k-1)/2 - 1) + 4) = 6k^2 - 14k.$$

Hence using Construction 2.3 establishes the bound  $v \geq 3m_{\max} + k - 3$  in Case (i). Note that the exceptional cases  $v \neq 231k + m, 234k + m, 237k + m$  do not affect the bound in general as  $3m_{\max} + k - 3 > 237k + m_{\max}$  if  $k \geq 23$ .

Similarly, for  $k \equiv 5 \pmod{12}$  with  $a = 1$  or  $4$  and  $s = 0, 1, \dots, (k-1)/4 - 1$  we get all residues modulo  $3(k-1)$  which are congruent to  $5$  or  $8 \pmod{12}$ . Again using these representative MRDs( $\{4, k\}, m_s$ ) with

$$m_{\max} = k(12s_{\max} + a_{\max}) = k(12((k-1)/4 - 1) + 4) = 3k^2 - 11k$$

in Construction 2.3 yields the bound in Case (ii). Again  $k \geq 53$  implies that  $3m_{\max} + k - 3 > 237k + m_{\max}$ , so the exceptional cases listed in the construction do not need to be considered.  $\square$

In view of the lemmas above it remains to investigate  $k = 5, 8, 11, 14, 17, 20, 29, 32, 41$ . The closures of  $K = \{4, 5\}$  and  $K = \{4, 8\}$  are almost completely known, see [5,3,22,1], so we just need to trace back the constructions and see which of them ensure that the designs constructed contain both blocks of size  $4$  and  $5$  or  $8$ , respectively.

**Lemma 2.6.** *If  $v \equiv 5, 8 \pmod{12}$ ,  $v \geq 17$ , then there is a mandatory representation design  $\text{MRD}(\{4, 5\}, v)$ .*

**Proof.**  $B(\{4, 5\}) = \mathbb{N}_{0,1 \pmod{4}} \setminus \{8, 9, 12\}$ , thus if  $v \equiv 5, 8 \pmod{12}$ ,  $v \geq 17$ , then  $v \in B(\{4, 5\})$ . Moreover, if  $v \not\equiv 5, 41 \pmod{60}$ , then the necessary conditions imply that  $v \notin B(\{4\}), B(\{5\})$  and, therefore, there is an  $\text{MRD}(\{4, 5\}, v)$ . Now, it is easily seen that each  $v \equiv 5, 41 \pmod{60}$ ,  $v \geq 65$  has a representation  $v = 4g + a$  with  $g \equiv 0, 1 \pmod{4}$ ,  $g \geq 16$ ,  $a \in \{1, 5, 13\}$ . Take a transversal design  $\text{TD}(5, g)$  which exists for all  $g \geq 11$  (see [10]), delete all but  $a$  points from the last group and fill in groups by a  $\text{PBD}(\{4, 5\}, g)$  or  $\text{PBD}(\{4, 5\}, a)$  to obtain an  $\text{MRD}(\{4, 5\}, v = 4g + a)$ . Noting that by Lemma 2.1(b.ii) there exists an  $\text{MRD}(\{4, 5\}, 41)$  completes the proof.  $\square$

**Lemma 2.7.** *If  $v \equiv 5, 8 \pmod{12}$ ,  $v \geq 104$ , then there is a mandatory representation design  $\text{MRD}(\{4, 8\}, v)$ .*

**Proof.**  $B(\{4, 8\}) \supseteq \mathbb{N}_{0,1 \pmod{4}} \setminus \{5, 9, 12, 17, 20, 21, 24, 33, 41, 44, 45, 48, 53, 60, 65, 69, 77, 89, 101\}$ , thus if  $v \equiv 5, 8 \pmod{12}$ ,  $v \geq 104$ ,  $v \not\equiv 8, 113 \pmod{168}$  then  $v \in B(\{4, 8\})$  but  $v \notin B(\{4\}), B(\{8\})$  and, therefore, there is an  $\text{MRD}(\{4, 8\}, v)$ . Each  $v \equiv 8 \pmod{168}$ ,  $v \geq 104$  has a representation  $v = 8u$ , where  $u \equiv 1 \pmod{3}$  and each  $v \equiv 113 \pmod{168}$  has a representation  $v = 7u + 1$ , where  $u \equiv 4 \pmod{12}$ , so Lemma 2.1(a.i), (b.i) provides in each case an  $\text{MRD}(\{4, 8\}, v)$ .  $\square$

So far we have not used 4-GDDs of type  $g^u m^1$  with  $g \equiv 1, 5 \pmod{6}$  from Lemma 1.5 as these give in general worse bounds compared to the bounds we already have. But there are fewer possible exceptions, so these GDDs are useful for small  $k$ .

**Construction 2.8.** *Let  $k \equiv 5 \pmod{6}$ ,  $k \neq 11, 17$  and suppose there is a  $\text{PBD}(\{4, k\}; m)$  with  $m \equiv 2 \pmod{3}$ . Then there is a mandatory representation design  $\text{MRD}(\{4, k\}; v)$  for all  $v \geq 3m + k + 3$  with  $v \equiv m \pmod{12k}$ ,  $v \neq 12k + m, 24k + m, 72k + m, 120k + m, 168k + m$ .*

**Proof.** Use a 4-GDD of type  $k^u m^1$  which exists by Lemma 1.5 for all  $u \equiv 0 \pmod{12}$ ,  $u \neq 12, 24, 72, 120, 168$ ,  $u \geq (2m + 3)/k + 1$ , consider groups of size  $k$  to be blocks and fill the group of size  $m$  by the  $\text{PBD}(\{4, k\}; m)$  to produce a  $\text{PBD}(\{4, k\}; v = uk + m)$ . Clearly,  $v \equiv m \pmod{12k}$  and we get a PBD for each such  $v$  with  $v \geq \lceil (2m + 3)/k + 1 \rceil_{0,12k} m \geq 3m + k + 3$  with the five exceptions listed. Since there is more than one group in the 4-GDD there are blocks of size 4 and  $k$ . So the resulting PBD is indeed an  $\text{MRD}(\{4, k\}; v)$  as desired.  $\square$

**Construction 2.9.** *Let  $k \equiv 2 \pmod{6}$ ,  $k \geq 8$  and suppose there is a  $\text{PBD}(\{4, k\}; m)$  with  $m \equiv 2 \pmod{3}$ . Then there is a mandatory representation design  $\text{MRD}(\{4, k\}; v)$  for all  $v \geq 3m + k$  with  $v \equiv m \pmod{12(k-1)}$ ,  $v \neq 12(k-1) + m, 24(k-1) + m, 72(k-1) + m, 120(k-1) + m, 168(k-1) + m$ .*

**Proof.** Take a 4-GDD of type  $(k-1)^u (m-1)^1$  which exists by Lemma 1.5 for all  $u \equiv 0 \pmod{12}$ ,  $u \neq 12, 24, 72, 120, 168$ ,  $u \geq (2(m-1) + 3)/(k-1) + 1$ , adjoin one infinite point and fill in the groups together with the infinite point by blocks of size  $k$  or the  $\text{PBD}(\{4, k\}; m)$  to obtain a  $\text{PBD}(\{4, k\}; v = u(k-1) + m)$ . Obviously,  $v \equiv m \pmod{12(k-1)}$  and  $v \geq \lceil (2(m-1) + 3)/(k-1) + 1 \rceil_{0,12(k-1)} (k-1) + m \geq 3m + k$ . Again, if  $m$  is relatively small we can have five possible exceptions. Note that the construction method ensures that there are blocks of size 4 and  $k$  and, therefore, the PBD constructed is an MRD.  $\square$

**Lemma 2.10.** *If  $v \equiv 2 \pmod{3}$ ,  $v \geq 2492$ ,  $v \neq 2513, 2516, 2546, 2585, 2615, 2618, 2645, 2648, 2678, 2717, 2747, 2750, 2777, 2780, 2810, 2849, 2879, 2882, 2909, 2912, 2942$ , then there is a mandatory representation design  $\text{MRD}(\{4, 11\}, v)$ .*

**Proof.** Let  $M = \{11, 44, 143, 176, 275, 308, 407, 440, 539, 572\}$ . Lemma 2.1(a.ii) provides a  $\text{PBD}(\{4, 11\}, m)$  for all  $m \in M$  and  $M$  contains one element in each residue class modulo 30 that is equivalent to  $2 \pmod{3}$ . Hence, Construction 2.3 yields an  $\text{MRD}(\{4, 11\}, v)$  for all  $v \equiv 2 \pmod{3}$  with  $v \geq \max\{3m_{\max} + 11 - 3 = 1724, 192 \cdot 10 + m_{\max} = 2492\} = 2492$  and  $v \neq 231 \cdot 10 + M, 234 \cdot 10 + M, 237 \cdot 10 + M$ .  $\square$

**Lemma 2.11.** *If  $v \equiv 5, 11 \pmod{12}$ ,  $v \geq 3677$ ,  $v \neq 3683, 3755, 3797, 3839, 3911, 3953, 3995, 4067, 4109, 4151, 4223, 4265, 4307$ , or if  $v \equiv 2, 8 \pmod{12}$ ,  $v \geq 3206$ ,  $v \neq 3206, 3248$  then there is a mandatory representation design  $\text{MRD}(\{4, 14\}, v)$ .*



**Proof.** Let  $M = \{156t + 53 : t = 0, \dots, 6\}$ . Lemma 2.1(b.i) (using GDDs of type  $13^u$  with  $u \equiv 4 \pmod{12}$ ) provides a PBD( $\{4, 14\}, m$ ) for all  $m \in M$ . Here,  $M$  contains one element in each residue class modulo 42 that is equivalent to 5 mod 6. Hence, Construction 2.2 yields an MRD( $\{4, 14\}, v$ ) for all  $v \equiv 5, 11 \pmod{12}$  with  $v \geq \max\{3m_{\max} + 14 = 2981, 192 \cdot 14 + m_{\max} = 3677\} = 3677$  and  $v \neq 231 \cdot 14 + M, 234 \cdot 14 + M, 237 \cdot 14 + M$ .

Now let  $R = \{42t + 14 : t = 0, \dots, 25\}$ . Lemma 2.1(a.i) (using GDDs of type  $14^u$ ) provides a PBD( $\{4, 14\}, r$ ) for all  $r \in R$ . Note that  $R$  contains one element in each residue class modulo 156 that is equivalent to 2 mod 6. Therefore, Construction 2.9 yields an MRD( $\{4, 14\}, v$ ) for all  $v \equiv 2, 8 \pmod{12}$  with  $v \geq 3r_{\max} + 14 = 3206$  and  $v \neq 169 \cdot 13 + R$ .  $\square$

**Lemma 2.12.** *If  $v \equiv 5, 8 \pmod{12}$ ,  $v \geq 3752$ ,  $v \neq 3761, 3764, 3809, 3812, 3860, 3917, 3965, 3968, 4013, 4016, 4064, 4121, 4169, 4172, 4217, 4220, 4268, 4325, 4373, 4376, 4421, 4424, 4472$ , then there is a mandatory representation design MRD( $\{4, 17\}, v$ ).*

**Proof.** Let  $M = \{17, 68, 221, 272, 425, 476, 629, 680\}$ . Lemma 2.1(a.ii) (using GDDs of type  $17^u$ ) provides a PBD( $\{4, 17\}, m$ ) for all  $m \in M$  and  $M$  contains one element in each residue class modulo 48 that is equivalent to 5 mod 6. Hence, Construction 2.3 yields an MRD( $\{4, 17\}, v$ ) for all  $v \equiv 5, 8 \pmod{12}$  with  $v \geq \max\{3m_{\max} + 17 - 3 = 2054, 192 \cdot 16 + m_{\max} = 3752\} = 3752$  and  $v \neq 231 \cdot 16 + M, 234 \cdot 16 + M, 237 \cdot 16 + M$ .  $\square$

**Lemma 2.13.** *If  $v \equiv 5 \pmod{12}$ ,  $v \geq 4829$ ,  $v \neq 4925, 4985, 5045, 5153, 5213, 5273, 5381, 5441, 5501, 5609, 5669, 5729$  or if  $v \equiv 8 \pmod{12}$ ,  $v \geq 3320$ ,  $v \neq 3320, 3332, 3380, 3392, 3452, 3512, 3572, 3632, 3692, 3752, 3812, 3872, 3932, 3992, 4052, 4112, 4172, 4232, 4292$  then there is a mandatory representation design MRD( $\{4, 20\}, v$ ).*

**Proof.** Let  $M = \{228t + 77 : t = 0, \dots, 4\}$ . Lemma 2.1(b.i) (using GDDs of type  $19^u$  with  $u \equiv 4 \pmod{12}$ ) provides a PBD( $\{4, 20\}, m$ ) for all  $m \in M$ .  $M$  contains one element in each residue class modulo 60 that is equivalent to 5 mod 12. Hence, Construction 2.2 yields an MRD( $\{4, 20\}, v$ ) for all  $v \equiv 5 \pmod{12}$  with  $v \geq \max\{3m_{\max} + 20 = 2987, 192 \cdot 20 + m_{\max} = 4829\} = 4829$  and  $v \neq 231 \cdot 20 + M, 234 \cdot 20 + M, 237 \cdot 20 + M$ .

Now let  $R = \{60t + 20 : t = 0, \dots, 18\}$ . Lemma 2.1(a.i) (using GDDs of type  $20^u$ ) provides a PBD( $\{4, 20\}, r$ ) for all  $r \in R$ .  $R$  contains one element in each residue class modulo 228 that is equivalent to 8 mod 12. Hence, Construction 2.9 yields an MRD( $\{4, 20\}, v$ ) for all  $v \equiv 8 \pmod{12}$  with  $v \geq 3r_{\max} + 20 = 3320$  and  $v \neq 120 \cdot 19 + R, 169 \cdot 19 + R$ .  $\square$

**Lemma 2.14.** *If  $v \equiv 8 \pmod{12}$ ,  $v \geq 7580$ ,  $v \neq 7628, 7712, 7796, 7976, 8060, 8144, 8324, 8408, 8492, 8672, 8756, 8840$ , or if  $v \equiv 5 \pmod{12}$ ,  $v \geq 7175$ ,  $v \neq 7253$ , then there is a mandatory representation design MRD( $\{4, 29\}, v$ ).*

**Proof.** Let  $M = \{348t + 116 : t = 0, \dots, 6\}$ . Lemma 2.1(a.ii) (using GDDs of type  $29^u$  with  $u \equiv 4 \pmod{12}$ ) provides a PBD( $\{4, 29\}, m$ ) for all  $m \in M$  and  $M$  contains one element in each residue class modulo 84 that is equivalent to 8 mod 12. Hence, Construction 2.3 yields an MRD( $\{4, 29\}, v$ ) for all  $v \equiv 8 \pmod{12}$  with  $v \geq \max\{3m_{\max} + 29 - 3 = 6638, 192 \cdot 28 + m_{\max} = 7580\} = 7580$  and  $v \neq 231 \cdot 28 + M, 234 \cdot 28 + M, 237 \cdot 28 + M$ .

Now let  $R = \{84t + 29 : t = 0, \dots, 28\}$ . Lemma 2.1(b.ii) (using GDDs of type  $28^u$ ) provides a PBD( $\{4, 29\}, r$ ) for all  $r \in R$  and  $R$  contains a representative element in each residue class modulo 384 that is equivalent to 5 mod 12. Hence, Construction 2.8 yields an MRD( $\{4, 29\}, v$ ) for all  $v \equiv 5 \pmod{12}$  with  $v \geq 3r_{\max} + 29 + 3 = 7175$ , and  $v \neq 168 \cdot 29 + R$ .  $\square$

**Lemma 2.15.** *If  $v \equiv 5 \pmod{12}$ ,  $v \geq 8873$ ,  $v \neq 9005, 9101, 9197, 9377, 9473, 9569, 9749, 9845, 9941, 10121, 10217, 10313$  or if  $v \equiv 8 \pmod{12}$ ,  $v \geq 8768$ , then there is a mandatory representation design MRD( $\{4, 32\}, v$ ).*

**Proof.** Let  $M = \{372t + 125 : t = 0, \dots, 7\}$ . Lemma 2.1(b.i) (using GDDs of type  $31^u$  with  $u \equiv 4 \pmod{12}$ ) provides a PBD( $\{4, 32\}, m$ ) for all  $m \in M$  and  $M$  contains one element in each residue class modulo 96 that is equivalent to 5 mod 12. Hence, Construction 2.2 yields an MRD( $\{4, 32\}, v$ ) for all  $v \equiv 5 \pmod{12}$  with  $v \geq \max\{3m_{\max} + 32 = 8312, 192 \cdot 32 + m_{\max} = 8873\} = 8873$  and  $v \neq 231 \cdot 32 + M, 234 \cdot 32 + M, 237 \cdot 32 + M$ .

Now let  $R = \{96t + 32 : t = 0, \dots, 30\}$ . Lemma 2.1(a.i) (using GDDs of type  $32^u$ ) provides a PBD( $\{4, 32\}, r$ ) for all  $r \in R$  and  $R$  contains one element in each residue class modulo 372 that is equivalent to 8 mod 12. Hence, Construction 2.9 yields an MRD( $\{4, 32\}, v$ ) for all  $v \equiv 8 \pmod{12}$  with  $v \geq 3r_{\max} + 32 = 8768$ .  $\square$

**Lemma 2.16.** *If  $v \equiv 5, 8 \pmod{12}$ ,  $v \geq 13814$ ,  $v \neq 13832, 13952, 14072$ , then there is a mandatory representation design  $\text{MRD}(\{4, 41\}, v)$ .*

**Proof.** Let  $M = \{492t + 41, 492t + 164 : t = 0, \dots, 9\}$ . Lemma 2.1 (b.i) provides a  $\text{PBD}(\{4, 41\}, m)$  for all  $m \in M$ .  $M$  contains one element in each residue class modulo 120 that is equivalent to 5 or 8 mod 12. Therefore, Construction 2.3 yields an  $\text{MRD}(\{4, 41\}, v)$  for all  $v \equiv 5, 8 \pmod{12}$  with  $v \geq \max\{3m_{\max} + 41 - 3 = 13814, 192 \cdot 40 + m_{\max} = 12272\} = 13814$  and  $v \neq 231 \cdot 40 + M, 234 \cdot 40 + M, 237 \cdot 40 + M$ . The latter inequality gives a list of five possible exceptions with  $v \geq 13814$ . We can delete  $v = 13829, 12949$  from that list as there are a 4-GDD of type  $41^{228}4481^1$ , a 4-GDD of type  $41^{228}4601^1$  (Lemma 1.5) and an  $\text{MRD}(\{4, 41\}, 4481)$  and an  $\text{MRD}(\{4, 41\}, 4601)$  (Lemma 2.1(b.ii)). Thus filling in groups yields the desired MRDs leaving three possible exceptions,  $v = 13832, 13952, 14072$ .  $\square$

## 2.2. $v \equiv 0 \pmod{3}$ , $\gamma_k \equiv 2 \pmod{3}$

Now, we turn our attention to MRDs on  $v \equiv 0 \pmod{3}$  points where  $\gamma_k$  needs to be congruent to 2 mod 3. The basic idea is to take a modified group divisible design and to construct on each group and on each hole an MRD with  $\gamma_k \equiv 1 \pmod{3}$  which provides, as every point occurs in exactly one group and exactly one hole, an MRD with  $\gamma_k \equiv 2 \pmod{3}$ . But first we state a more general construction using modified group divisible designs.

**Construction 2.17.** *Let  $k \equiv 2 \pmod{3}$ ,  $k \geq 5$ . If there is a  $\text{PBD}(\{4, k\}; v)$ , then there is a mandatory representation design  $\text{MRD}(\{4, k\}; (k-1)v+1)$ . If there is a  $\text{PBD}(\{4, k\}; v)$  with  $v \equiv 2 \pmod{3}$ , then there is a mandatory representation design  $\text{MRD}(\{4, k\}; k(v-1)+1)$ .*

**Proof.** Clearly,  $((k-1)-1)(v-1) \equiv 0 \pmod{3}$  and therefore Lemma 1.6 implies that there is a 4-MGDD of type  $(k-1)^v$ . So take that 4-MGDD and fill each hole by the  $\text{PBD}(\{4, k\}; v)$ . Furthermore, adjoin a new point to the point set and replace each group and the new point by a  $k$ -block to produce an MRD with blocks of size 4 and  $k$  on  $(k-1)v+1$  points. Similarly, if in addition  $v \equiv 2 \pmod{3}$ , then  $(k-1)((v-1)-1) \equiv 0 \pmod{3}$ , which implies that there exists a 4-MGDD of type  $k^{v-1}$ . Again, we adjoin a new point and replace now each hole and this new point by the  $\text{PBD}(\{4, k\}; v)$ . If we consider all groups to be blocks of size  $k$ , then we get the desired  $\text{MRD}(\{4, k\}; k(v-1)+1)$ .  $\square$

Using this construction together with designs from Lemma 2.1 we obtain the following three corollaries.

**Corollary 2.18.** *Let  $k \equiv 2 \pmod{3}$ ,  $k \geq 5$ . There is a mandatory representation design  $\text{MRD}(\{4, k\}; k(k-1)u+1)$  for all  $u \equiv 1 \pmod{3}$ .*

**Proof.** Start with a single block of size  $k$  as a (trivial) PBD in Construction 2.17 to obtain an  $\text{MRD}(\{4, k\}; k(k-1)+1)$ . Then use this design to fill in the groups of a 4-GDD of type  $(k(k-1))^u$  with a point at infinity adjoined which exists for all  $u \equiv 1 \pmod{3}$  by Lemma 1.3 since  $k(k-1) \equiv 2 \pmod{6}$ . The number of points in the MRD constructed is  $k(k-1)u+1$  as desired.  $\square$

**Corollary 2.19.** *Let  $k \equiv 2 \pmod{3}$ ,  $k \geq 5$ . There is a mandatory representation design  $\text{MRD}(\{4, k\}; k(ku-1)+1)$*

- (i) *if  $k \equiv 2, 8 \pmod{12}$  and  $u \equiv 1 \pmod{3}$ ; or*
- (ii) *if  $k \equiv 5, 11 \pmod{12}$  and  $u \equiv 1, 4 \pmod{12}$ .*

**Proof.** An MRD with  $k(k-1)+1$  points is already constructed in Corollary 2.18 for all  $k \equiv 2 \pmod{3}$ ,  $k \geq 5$ , so we only need to consider  $u \geq 4$ . For that purpose take an MRD with  $v = ku$  from Lemma 2.1(a) for which clearly  $v \equiv 2 \pmod{3}$  holds. Thus applying Construction 2.17 yields the desired  $\text{MRD}(\{4, k\}; k(ku-1)+1)$ .  $\square$

**Corollary 2.20.** *Let  $k \equiv 2 \pmod{3}$ ,  $k \geq 5$ . There is a mandatory representation design  $\text{MRD}(\{4, k\}; (k-1)^2u+k)$*

- (i) *if  $k \equiv 2, 8 \pmod{12}$  and  $u \equiv 1, 4 \pmod{12}$ ; or*
- (ii) *if  $k \equiv 5, 11 \pmod{12}$  and  $u \equiv 1 \pmod{3}$ .*

**Proof.** For  $u = 1$  take an MRD with  $v = k(k-1)+1$  from Corollary 2.18. For  $u \geq 4$  apply Construction 2.17 with an  $\text{MRD}(\{4, k\}; (k-1)u+1)$  constructed in Lemma 2.1(b) to obtain an MRD on  $(k-1)((k-1)u+1)+1 = (k-1)^2u+k$  points.  $\square$



We remark that for the number of points  $v = k(k-1)u + 1$ ,  $v = k(ku-1) + 1$  or  $v = (k-1)^2u + k$  of the MRDs constructed above  $v \equiv 0 \pmod 3$  holds, and that every point is contained in either 2,  $k$ ,  $u+1$ ,  $2u$ ,  $ku$  or  $(k-1)u+1$  blocks of size  $k$ . Hence,  $\gamma_k \equiv 2 \pmod 3$  as desired.

Similarly as in the case  $v \equiv 2 \pmod 3$  one can construct an infinite sequence of mandatory representation designs from just one ingredient MRD with the property that if  $\gamma_k \equiv 2 \pmod 3$  for each point in the ingredient MRD, then also in the resulting MRD  $\gamma_k \equiv 2 \pmod 3$  for all points.

**Construction 2.21.** Let  $k \equiv 2 \pmod 3$  and suppose there is an MRD( $\{4, k\}; m$ ) with  $m \equiv 0 \pmod 3$ . Then there is a mandatory representation design MRD( $\{4, k\}; v$ ) for all  $v \geq 3m + k(k-1) + 4$  with  $v \equiv m \pmod{4(k(k-1)+1)}$ ,  $v \neq 8(k(k-1)+1) + m$ ,  $12(k(k-1)+1) + m$ .

**Proof.** If  $k \equiv 2 \pmod 3$ , then  $k(k-1)+1 \equiv 3 \pmod 6$  and so by Lemma 1.5 there is a 4-GDD of type  $(k(k-1)+1)^u m^1$  for all  $u \equiv 0 \pmod 4$ ,  $u \neq 8, 12$ ,  $u \geq (2m+3)/(k(k-1)+1) + 1$ . Replacing groups of size  $k(k-1)+1$  by an MRD( $\{4, k\}; k(k-1)+1$ ) which exists by Corollary 2.18 and the group of size  $m$  by the MRD( $\{4, k\}; m$ ) produces an MRD( $\{4, k\}; v = u(k(k-1)+1) + m$ ). Thus we get an MRD for all  $v \equiv m \pmod{4(k(k-1)+1)}$ , where  $v \geq \lceil (2m+3)/(k(k-1)+1) + 1 \rceil_{0;4} (k(k-1)+1) + m \geq 3m + k(k-1) + 4$ , with the exception of  $v = 8(k(k-1)+1) + m$  or  $v = 12(k(k-1)+1) + m$ .  $\square$

In what follows we want to apply Construction 2.21. If we are able to provide a representative MRD( $\{4, k\}; m_t$ ) with  $m_t \equiv 3t \pmod{4(k(k-1)+1)}$  for each  $t = 0, 1, \dots, 4(k(k-1)+1)/3 - 1$  if  $k \equiv 2, 11 \pmod{12}$ ; or with  $m_t \equiv 6t + 3 \pmod{4(k(k-1)+1)}$  for each  $t = 0, 1, \dots, 4(k(k-1)+1)/6 - 1$  if  $k \equiv 5, 8 \pmod{12}$ , then we have established the existence of an MRD( $\{4, k\}; v$ ) for all  $v \geq 3m_{\max} + g + 3$ , where  $m_{\max} = \max\{m_t\}$ . We remark that the designs resulting from Construction 2.21 lie in the same residue class modulo 12 as the second ingredient MRD( $\{4, k\}; m$ ). So we will need to consider different types of ingredient MRDs to obtain the desired designs in each residue class modulo 12. This will be done in the next three lemmas.

**Lemma 2.22.** Let  $k \equiv 2 \pmod 3$  and  $v \equiv 3, 9 \pmod{12}$ . There exists a mandatory representation design MRD( $\{4, k\}; v$ )

- (i) if  $k \equiv 2, 11 \pmod{12}$ ,  $v \equiv 3, 9 \pmod{12}$  and  $v \geq 6k^4 - 12k^3 + 25k^2 - 19k + 7$ ; or
- (ii) if  $k \equiv 5, 8 \pmod{12}$ ,  $v \equiv 9 \pmod{12}$  and  $v \geq 3k^4 - 6k^3 + 10k^2 - 7k + 7$ .

**Proof.** For  $k \equiv 2, 11 \pmod{12}$  it suffices to provide representative MRD( $\{4, k\}; m_t$ ) with  $m_t \equiv 12t + 3 \pmod{4(k(k-1)+1)}$  and  $m_t \equiv 12t + 9 \pmod{4(k(k-1)+1)}$  for each  $t = 0, 1, \dots, (k(k-1)+1)/3 - 1$  to obtain the desired bounds. These representative MRDs are taken from Corollary 2.18: an MRD( $\{4, k\}; m_s$ ) with  $m_s = k(k-1)u_s + 1$  exists for all  $u_s = 12s + a$ , where  $s \in \mathbb{N}$ ,  $a \in \{1, 4, 7, 10\}$ . Then

$$m_s = k(k-1)(12s + a) + 1 = 3s(4(k(k-1)+1)) - 12s + ak(k-1) + 1$$

and thus  $m_s \equiv -12s + ak(k-1) + 1 \pmod{4(k(k-1)+1)}$ . Since  $\gcd(12, 4(k(k-1)+1)) = 12$  it is easy to check that with  $a = 1$  or  $a = 7$  and  $s = 0, 1, \dots, (k(k-1)-2)/6$  we get all residues modulo  $4(k(k-1)+1)$  which are congruent to 3 modulo 12. Moreover, with  $a = 4$  or  $a = 10$  and  $s = 0, 1, \dots, (k(k-1)-2)/6$  we get all residues modulo  $4(k(k-1)+1)$  which are congruent to 9 modulo 12. The largest representative MRD has order

$$\begin{aligned} m_{\max} &= k(k-1)(12s_{\max} + a_{\max}) + 1 \\ &= k(k-1)(12(k(k-1)-2)/6 + 10) + 1 \\ &= 2k^2(k-1)^2 + 6k(k-1) + 1. \end{aligned}$$

Hence using Construction 2.21 establishes the bound  $v \geq 3m_{\max} + g + 3$  in Case (i). Note that the exceptional case  $v \neq 8g + m$ ,  $12g + m$  does not affect the bound as  $3m_{\max} > 12g + m_{\max}$ .

Similarly, for  $k \equiv 5, 8 \pmod{12}$  it suffices to present representative MRD( $\{4, k\}; m_t$ ) with  $m_t \equiv 12t + 9 \pmod{4(k(k-1)+1)}$  for each  $t = 0, 1, \dots, (k(k-1)+1)/3 - 1$  to obtain the desired bounds. With  $u = 12s + a$ ,  $a = 1, 4, 7$  or  $10$  and  $s = 0, 1, \dots, (k(k-1)-8)/12$  we get all residues modulo  $4(k(k-1)+1)$  which are congruent to 9 modulo 12. Again using these representative MRDs( $\{4, k\}; m_s$ ) with

$$m_{\max} = k(k-1)(12(k(k-1)-8)/12 + 10) + 1 = k^2(k-1)^2 + 2k(k-1) + 1$$

in Construction 2.21 yields the bound in Case (ii).  $\square$

**Lemma 2.23.** Let  $k \equiv 5, 11 \pmod{12}$  and  $v \equiv 0, 6 \pmod{12}$ . There exists a mandatory representation design  $\text{MRD}(\{4, k\}; v)$

- (i) if  $k \equiv 5 \pmod{12}$ ,  $v \equiv 0 \pmod{12}$  and  $v \geq 4k^4 - 5k^3 + 43k^2 - 12k + 14$ ; or
- (ii) if  $k \equiv 11 \pmod{12}$ ,  $v \equiv 0, 6 \pmod{12}$  and  $v \geq 8k^4 - 10k^3 + 49k^2 - 14k + 15$ .

**Proof.** First, let  $k \equiv 5 \pmod{12}$ . It suffices to provide representative  $\text{MRD}(\{4, k\}, m_t)$  with  $m_t \equiv 12t \pmod{4(k(k-1)+1)}$  for each  $t = 0, 1, \dots, (k(k-1)+1)/3 - 1$ . There is an  $\text{MRD}(\{4, k\}; k(4k-1)+1)$  by Corollary 2.19 with  $k(4k-1)+1 \equiv 0 \pmod{12}$ . Lemma 1.3 implies that there exists a 4-GDD of type  $(k(4k-1)+1)^s$  for all  $s \geq 4$  and, therefore, an  $\text{MRD}(\{4, k\}, m_s = (k(4k-1)+1)s)$ . It is easy to check that

$$m_s = ((k(4k-1)+1)s) = s(4(k(k-1)+1)) + 3(k-1)s$$

and thus  $m_s \equiv 3(k-1)s \pmod{4(k(k-1)+1)}$ . Clearly  $\gcd(3(k-1), 4(k(k-1)+1)) = 12$ , so it follows immediately that with  $s = 4, 5, \dots, (k(k-1)+10)/3$  we get all residues modulo  $4(k(k-1)+1)$  which are congruent to 0 modulo 12. The largest representative MRD has order

$$\begin{aligned} m_{\max} &= (k(4k-1)+1)s_{\max} \\ &= (k(4k-1)+1)((k(k-1)+10)/3) \\ &= \frac{1}{3}(4k^4 - 5k^3 + 42k^2 - 11k + 10). \end{aligned}$$

Hence using Construction 2.21 establishes the bound in Case (i).

Now, let  $k \equiv 11 \pmod{12}$ . It suffices to provide representative  $\text{MRD}(\{4, k\}, m_t)$  with  $m_t \equiv 12t$ , or  $12t + 6 \pmod{4(k(k-1)+1)}$  for each  $t = 0, 1, \dots, (k(k-1)+1)/3 - 1$ . In a similar way as in the discussion above one shows that there is an  $\text{MRD}(\{4, k\}, m_s = (k(4k-1)+1)s)$  for all  $s \geq 4$ . Note that  $k(4k-1)+1 \equiv 6 \pmod{12}$ , and hence if  $s$  is even and  $s = 0, 2, \dots, 2(k(k-1)+4)/3$ , then we get all residues modulo  $4(k(k-1)+1)$  which are congruent to 0 modulo 12, while, if  $s$  is odd and  $s = 1, 3, \dots, 2(k(k-1)+4)/3 + 1$ , then we get all residues modulo  $4(k(k-1)+1)$  which are congruent to 6 modulo 12. Again using these representative MRDs  $\{4, k\}, m_s$  with

$$m_{\max} = (k(4k-1)+1)(2(k(k-1)+4)/3 + 1) = \frac{1}{3}(8k^4 - 10k^3 + 48k^2 - 13k + 11)$$

in Construction 2.21 yields the bound in Case (ii).  $\square$

**Lemma 2.24.** Let  $k \equiv 2, 8 \pmod{12}$  and  $v \equiv 0, 6 \pmod{12}$ . There exists a mandatory representation design  $\text{MRD}(\{4, k\}; v)$

- (i) if  $k \equiv 2 \pmod{12}$ ,  $v \equiv 0, 6 \pmod{12}$  and  $v \geq 8k^4 - 22k^3 + 67k^2 - 86k + 48$ ; or
- (ii) if  $k \equiv 8 \pmod{12}$ ,  $v \equiv 0 \pmod{12}$  and  $v \geq 4k^4 - 11k^3 + 52k^2 - 75k + 44$ .

**Proof.** First, let  $k \equiv 8 \pmod{12}$ . It suffices to provide representative MRDs  $\{4, k\}, m_t$  with  $m_t \equiv 12t \pmod{4(k(k-1)+1)}$  for each  $t = 0, 1, \dots, (k(k-1)+1)/3 - 1$ . There is an  $\text{MRD}(\{4, k\}; 4(k-1)^2 + k)$  by Corollary 2.20 with  $4(k-1)^2 + k \equiv 0 \pmod{12}$ . Lemma 1.3 implies that there exists a 4-GDD of type  $(4(k-1)^2 + k)^s$  for all  $s \geq 4$  and, therefore, an  $\text{MRD}(\{4, k\}, m_s = (4(k-1)^2 + k)s)$ . It is easy to check that

$$m_s = (4(k-1)^2 + k)s = s(4(k(k-1)+1)) - 3ks$$

and thus  $m_s \equiv -3ks \pmod{4(k(k-1)+1)}$ . Clearly  $\gcd(3k, 4(k(k-1)+1)) = 12$ , so it follows immediately that with  $s = 4, 5, \dots, (k(k-1)+10)/3$  we get all residues modulo  $4(k(k-1)+1)$  which are congruent to 0 modulo 12. The largest representative MRD has order

$$\begin{aligned} m_{\max} &= (4(k-1)^2 + k)s_{\max} \\ &= (4(k-1)^2 + k)((k(k-1)+10)/3) \\ &= \frac{1}{3}(4k^4 - 11k^3 + 51k^2 - 74k + 40). \end{aligned}$$

Hence using Construction 2.21 establishes the bound in Case (ii).

Now, let  $k \equiv 2 \pmod{12}$ . It suffices to provide representative MRDs( $\{4, k\}, m_t$ ) with  $m_t \equiv 12t$ , or  $12t + 6 \pmod{4(k(k-1)+1)}$  for each  $t = 0, 1, \dots, (k(k-1)+1)/3 - 1$ . As described above there is an MRD( $\{4, k\}, m_s = (4(k-1)^2 + k)s$ ) for all  $s \geq 4$ . Note that  $4(k-1)^2 + k \equiv 6 \pmod{12}$ , and hence if  $s$  is even and  $s = 0, 2, \dots, 2(k(k-1)+4)/3$ , then we get all residues modulo  $4(k(k-1)+1)$  which are congruent to 0 modulo 12, while, if  $s$  is odd and  $s = 1, 3, \dots, 2(k(k-1)+4)/3 + 1$ , then we get all residues modulo  $4(k(k-1)+1)$  which are congruent to 6 modulo 12. Again using these representative MRDs( $\{4, k\}, m_s$ ) with

$$m_{\max} = (4(k-1)^2 + k)(2(k(k-1)+4)/3 + 1) = \frac{1}{3}(8k^4 - 22k^3 + 66k^2 - 85k + 44)$$

in [Construction 2.21](#) yields the bound in Case (i).  $\square$

### 2.3. $v \equiv 1 \pmod{3}$ , $\gamma_k \equiv 0 \pmod{3}$

Using MRDs with  $\gamma_k \equiv 2 \pmod{3}$  from the previous subsection we are now able in conjunction with [Construction 2.17](#) to establish the existence of some MRDs with  $\gamma_k \equiv 0 \pmod{3}$  for each  $k \equiv 2 \pmod{3}$ . These MRDs are then used to fill groups of appropriate 4-GDDs.

**Corollary 2.25.** *Let  $k \equiv 2 \pmod{3}$ ,  $k \geq 5$ . There is a mandatory representation design  $\text{MRD}(\{4, k\}; k(k-1)^2u + k)$*

- (i) *if  $k \equiv 2, 8 \pmod{12}$  and  $u \equiv 1 \pmod{3}$ ,  $u \geq 4$ ; or*
- (ii) *if  $k \equiv 5, 11 \pmod{12}$  and  $u \equiv 1, 4 \pmod{12}$ ,  $u \geq 4$ .*

**Proof.** Use an MRD( $\{4, k\}; k(k-1)u + 1$ ) constructed in [Corollary 2.18](#) as ingredient PBD in [Construction 2.17](#) to obtain an MRD( $\{4, k\}; (k-1)(k(k-1)u + 1) + 1$ ).  $\square$

**Construction 2.26.** *Let  $t \equiv 1 \pmod{3}$  and suppose there is an MRD( $\{4, k\}; t$ ). Then there is a mandatory representation design  $\text{MRD}(\{4, k\}; v)$  for all  $v \geq 3t + 1$*

- (i) *with  $v \equiv 1, 4 \pmod{12}$  if  $t \equiv 1, 4 \pmod{12}$ ; or*
- (ii) *with  $v \equiv 7, 10 \pmod{12}$  if  $t \equiv 0, 9 \pmod{12}$ .*

**Proof.** Use a 4-GDD of type  $t^1 1^{v-t}$  from [Lemma 1.4](#) and replace the group of size  $t$  by the MRD( $\{4, k\}; t$ ).  $\square$

**Lemma 2.27.** *Let  $k \equiv 2 \pmod{3}$  and  $v \equiv 1 \pmod{3}$ . There exists a mandatory representation design  $\text{MRD}(\{4, k\}; v)$*

- (i) *if  $k \equiv 2, 5, 8 \pmod{12}$ ,  $v \equiv 1, 4 \pmod{12}$  and  $v \geq 3k(k-1)^2 + 3k + 1$ ; or*
- (ii) *if  $k \equiv 11 \pmod{12}$ ,  $v \equiv 1, 4 \pmod{12}$  and  $v \geq 12k(k-1)^2 + 12k - 8$ ; or*
- (iii) *if  $k \equiv 2 \pmod{12}$ ,  $v \equiv 7, 10 \pmod{12}$  and  $v \geq 12k(k-1)^2 + 3k + 1$ ; or*
- (iv) *if  $k \equiv 11 \pmod{12}$ ,  $v \equiv 7, 10 \pmod{12}$  and  $v \geq 3k(k-1)^2 + 3k + 1$ .*

**Proof.** Start with an MRD( $\{4, k\}; k(k-1)^2 + k$ ) from [Corollary 2.25](#) and use it as ingredient in [Construction 2.26](#). If  $k \equiv 2, 5, 8 \pmod{12}$  the number of points  $t = k(k-1)^2 + k$  is congruent 1 or 4 modulo 12, so an MRD for all  $v \geq 3(k(k-1)^2 + k) + 1$ ,  $v \equiv 1, 4 \pmod{12}$  is produced which gives the bound for Case (i), while, if  $k \equiv 11 \pmod{12}$ , we have  $t \equiv 7 \pmod{12}$  and, therefore,  $v \equiv 7, 10 \pmod{12}$  (Case (iv)).

For  $k \equiv 11 \pmod{12}$  we continue by filling in the MRD( $\{4, k\}; k(k-1)^2 + k$ ) into the groups of a 4-GDD of type  $(k(k-1)^2 + k)^4$  and get an MRD on  $t = 4(k(k-1)^2 + k)$ , where  $t \equiv 4 \pmod{12}$ . Thus, if used with [Construction 2.26](#) MRD( $\{4, k\}; v$ )s for all  $v \geq 12(k(k-1)^2 + k) + 1$ ,  $v \equiv 1, 4 \pmod{12}$  are obtained (Case (ii)).

Finally, take for  $k \equiv 2 \pmod{12}$  an MRD( $\{4, k\}; 4k(k-1)^2 + k$ ) which is obtained from [2.25](#) by setting  $u = 4$ . Here,  $t = 4k(k-1)^2 + k \equiv 10 \pmod{12}$ , so using again [Construction 2.26](#) yields an MRD( $\{4, k\}; v$ )s for all  $v \geq 12k(k-1)^2 + 3k + 1$ ,  $v \equiv 1, 4 \pmod{12}$  and establishes the bound in Case (iii).  $\square$

## 2.4. Main result

We summarize the main result of the section which is a combination of Lemmas 2.4–2.7, 2.10, 2.11, 2.11, 2.13–2.16, 2.22–2.24 and 2.27.

**Theorem 2.28.** *Let  $k \equiv 2 \pmod{3}$ ,  $k \geq 5$ . There exists a mandatory representation design  $\text{MRD}(\{4, k\}; v)$*

- (i) *with  $v \equiv 1 \pmod{3}$ , if*
  - (a)  $k \equiv 2, 5, 8 \pmod{12}$ ,  $v \equiv 1, 4 \pmod{12}$ , or  $k \equiv 11 \pmod{12}$ ,  $v \equiv 7, 10 \pmod{12}$ , and  $v \geq 3k(k-1)^2 + 3k + 1$ , or
  - (b)  $k \equiv 2 \pmod{12}$ ,  $v \equiv 7, 10 \pmod{12}$ , or  $k \equiv 11 \pmod{12}$ ,  $v \equiv 1, 4 \pmod{12}$ , and  $v \geq 12k(k-1)^2 + 12k - 8$ ;
- (ii) *with  $v \equiv 2 \pmod{3}$ , if*
  - (a)  $k \equiv 5, 8 \pmod{12}$ ,  $k = 5, 8$  or  $k \geq 44$ ,  $v \equiv 5, 8 \pmod{12}$  and  $v \geq 9k^2 - 32k + 27$ , or
  - (b)  $k \equiv 2, 11 \pmod{12}$ ,  $k \geq 23$  and  $v \geq 18k^2 - 41k + 27$ , or
  - (c)  $k = 17, 20, 29, 32, 41$  and  $v > 192k + 3k^2 - 11k + 9$ , or
  - (d)  $k = 11, 14$  and  $v > 192k + 6k^2 - 14k + 9$ ;
- (iii) *with  $v \equiv 0 \pmod{3}$ , if*
  - (a)  $k \equiv 5, 8 \pmod{12}$ ,  $v \equiv 9 \pmod{12}$  and  $v \geq 3k^4 - 6k^3 + 10k^2 - 7k + 7$ , or
  - (b)  $k \equiv 2, 11 \pmod{12}$ ,  $v \equiv 3, 9 \pmod{12}$  and  $v \geq 6k^4 - 12k^3 + 25k^2 - 19k + 7$ , or
  - (c)  $k \equiv 5, 8 \pmod{12}$ ,  $v \equiv 0 \pmod{12}$  and  $v \geq 4k^4 - 5k^3 + 43k^2 - 12k + 14$ , or
  - (d)  $k \equiv 2, 11 \pmod{12}$ ,  $v \equiv 0, 6 \pmod{12}$  and  $v \geq 8k^4 - 10k^3 + 49k^2 - 14k + 15$ .

## 3. Conclusion

After having established upper bounds for the case  $k \equiv 2 \pmod{3}$ , it remains to close the gap between lower and upper bounds. We remark that the analogue problem for  $K = \{3, k\}$  is difficult and far from being completely solved; see [15] for recent advances.

Also, the determination of the MRD-closure in the case  $k \equiv 0 \pmod{3}$  is open. Here, the difficulties arise out of the fact that each point lies on either  $\gamma_k \equiv 0 \pmod{3}$  or  $\gamma_k \equiv 1 \pmod{3}$  blocks of size  $k$ . So in order to construct MRDs with  $\gamma_k \equiv 0 \pmod{3}$  we cannot use modified group divisible designs together with MRDs with  $\gamma_k \equiv 1$  or  $2 \pmod{3}$  since the latter simply do not exist. It would be of considerable interest to establish something like a *further* modified group divisible design, i.e. a design with three parallel classes of equal sized holes, as this would allow us to fill each parallel class just with MRDs with  $\gamma_k \equiv 1 \pmod{3}$ .

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